明治大学科学技術研究所紀要
31(7)：85－116，1992

Mem. Inst. Sci. Tech. Meiji Univ.
31(7)：85－116，1992

Symplectic Structure of CFT of Free Bosons

Kiyoshi Hayashi

Izumi Campus, Meiji University, Eifuku 1-9-1,
Suginami-ku, Tokyo, 168 Japan

Received October 2, 1992 ; Accepted February 1, 1993

Synopsis: We consider a system of free bosons (bosonic closed string model) which has the conformal and (abelian) gauge symmetry.
In this note we state Noether's Theorem explicitly, emphasizing that there is a possibility of the appearance of a cocycle term in the relation of Poisson brackets of Noether charges already in the classical level, and apply it to the system. The cocycle term appears in the case of the gauge symmetry, but does not in the conformal symmetry one. The reason of the difference is not yet understood.

§1 Introduction

Conformal Field Theory (CFT) is governed by the Virasoro algebra

$$\text{(1.1a)} \qquad \mathrm{Vir} = \bigoplus_{n \in \mathbf{Z}} \mathbf{C}\hat{L}_n \,\oplus\, \mathbf{C}\hat{c}$$

with $(m, n \in \mathbf{Z})$

$$\text{(1.1b)} \qquad [\hat{L}_m, \hat{L}_n] = (m-n)\,\hat{L}_{m+n} + \frac{\hat{c}}{12}\,(m^3 - m)\,\delta_{m+n,0}\ ,$$

$$\text{(1.1c)} \qquad [\hat{L}_m,\ \hat{c}\,] = 0\ .$$

A representation of Vir is constructed from the following Heisenberg algebra

$$\text{(1.2a)} \qquad \text{Heis} = \bigoplus_{n \in \mathbf{Z}} \mathbf{C}\hat{a}_n \oplus \mathbf{C}\hat{\hbar}$$

with $(m, n \in \mathbf{Z})$

$$\text{(1.2b)} \qquad [\hat{a}_m, \hat{a}_n] \;=\; m\,\delta_{m+n,0}\,\hat{\hbar}\,,$$

$$\text{(1.2c)} \qquad [\hat{a}_m, \hat{\hbar}] \;=\; 0\,,$$

as follows. See Lecture 2 in [4] for detail. We first represent Heis on the space B of polynomials in infinitely many variables :

$$\text{(1.3)} \qquad B = \mathbf{C}[t_1,\ t_2,\ \cdots]$$

as

$$\text{(1.4)} \qquad \hat{a}_n = \frac{\partial}{\partial t_n}\,, \quad \hat{a}_{-n} = n\,t_n \quad (\,n \in \mathbf{Z}_{>0}\,)\,, \quad \hat{a}_0 = \mu I\,, \quad \hat{\hbar} = I$$

where $I : B \to B$ is the identity mapping and $\mu \in \mathbf{R}$. This is the highest weight representation for a weight

$$\text{(1.5)} \qquad \hat{a}_0 \mapsto \mu, \quad \hat{\hbar} \mapsto 1\,,$$

considering $\mathbf{C}\hat{a}_0 \oplus \mathbf{C}\hat{\hbar}$ as a Cartan subalgebra, whose highest weight vector is 1 as a polynomial. The space B is regarded as the Fock space of a system of free bosons, whose vacuum is the 1.

We put (Sugawara construction, (2.9) in [4])

$$\text{(1.6)} \qquad \hat{L}_n = \frac{1}{2} \sum_{j \in \mathbf{Z}} :\hat{a}_{-j}\,\hat{a}_{j+n}:\,, \qquad n \in \mathbf{Z}$$

where : : means the normal ordering defined by

$$\text{(1.7)} \qquad :\hat{a}_i\,\hat{a}_j: \;= \begin{cases} \hat{a}_i\,\hat{a}_j & \text{if} \quad i \leq j\,, \\ \hat{a}_j\,\hat{a}_i & \text{if} \quad i > j\,. \end{cases}$$

Then we have (Prop. 2.3 in [4])

$$\text{(1.8)} \qquad [\hat{L}_m, \hat{L}_n] \;=\; (m-n)\,\hat{L}_{m+n} + \frac{1}{12}\,(m^3 - m)\,\delta_{m+n,0}\,.$$

This means $\hat{L}_n$'s given by (1.6) provide a representation of Vir in B with central charge $c = 1$ (c is a number for which $\hat{c}$ is represented as $\hat{c} = c\,I$). We also have (Lemma 2.2 in [4])

(1.9) $$[\hat{L}_m, \hat{a}_n] \;=\; -\,n\,\hat{a}_{m+n}\,, \quad m,n \in \mathbf{Z}\,.$$

The relation (1.6) is equivalent to the following one between field operators:

(1.10) $$\hat{T}(\theta) = \frac{1}{2} : \hat{J}(\theta)\,\hat{J}(\theta) :, \quad \theta \in S^1 = \mathbf{R}/2\pi\mathbf{Z}$$

where

(1.11) $$\hat{T}(\theta) = \sum_{n\in\mathbf{Z}} e^{-in\theta}\,\hat{L}_n\,,$$

(1.12) $$\hat{J}(\theta) = \sum_{n\in\mathbf{Z}} e^{-in\theta}\,\hat{a}_n\,.$$

We call (1.10) the *Sugawara relation.* (We do not care the convergence problem. The above expressions (1.11) and (1.12) are regarded as formal ones.)

We renumber the relations (1.8), (1.2b) with $\hat{\hbar} = I$ and (1.9) for later convenience : $(m,n \in \mathbf{Z})$

(1.13a) $$[\hat{L}_m, \hat{L}_n] \;=\; (m-n)\,\hat{L}_{m+n} + \frac{1}{12}\,(m^3 - m)\,\delta_{m+n,0}\,,$$

(1.13b) $$[\,\hat{a}_m, \hat{a}_n\,] \;=\; m\,\delta_{m+n,0}\,,$$

(1.13c) $$[\hat{L}_m, \hat{a}_n\,] \;=\; -\,n\,\hat{a}_{m+n}\,.$$

The group theoretical meaning of (1.13 a – c) is as follows. We consider the two groups

(1.14) $\mathcal{D} = \mathrm{Diff}^+ S^1 = \{\phi : S^1 \to S^1,$ orientation preserving diffeomorphism$\}$

whose group operation is given by the composition and

(1.15) $$\mathcal{N} = S^1\mathbf{R}^N = \{h : S^1 \to \mathbf{R}^N,\ C^\infty\text{-map}\}$$

with the operation of the pointwise addition in $\mathbf{R}^N$ (hereafter we take $N = 1$ for simplicity, but retain N to distinguish $\mathbf{R}^N$ from $\mathbf{R}$ as the space of τ, θ or x^+).

We let $\mathcal{D}$ and $\mathcal{N}$ act on the space

$$S^1\mathbf{R}^N = \{X : S^1 \to \mathbf{R}^N,\ C^\infty\text{-map}\} \tag{1.16}$$

by

$$(\phi \cdot X)(\theta) = X(\phi^{-1}(\theta)) \quad \text{for} \quad \phi \in \mathcal{D},\ X \in S^1\mathbf{R}^N\ , \tag{1.17}$$

$$(h \cdot X)(\theta) = X(\theta) + h(\theta) \quad \text{for} \quad h \in \mathcal{N},\ X \in S^1\mathbf{R}^N\ . \tag{1.18}$$

The action of the semi-direct product $\mathcal{D} \ltimes \mathcal{N}$ is given by

$$((\phi, h) \cdot X)(\theta) = X(\phi^{-1}(\theta)) + h(\phi^{-1}(\theta))\ , \tag{1.19}$$

then, in the operation form in $\mathcal{D} \ltimes \mathcal{N}$, we have

$$(\phi_1, h_1) \cdot (\phi_2, h_2) = (\phi_1 \cdot \phi_2, h_1 \circ \phi_2 + h_2)\ . \tag{1.20}$$

The Lie algebra of $\mathcal{D}$ is

$$\underline{\mathcal{D}} = S^1\mathbf{R} = \{\xi : S^1 \to \mathbf{R}\}\ , \tag{1.21}$$

whose Lie bracket is given by

$$[\xi, \eta] = \xi'\eta - \xi\eta'\ , \quad \xi, \eta\ \in\ \underline{\mathcal{D}}\ , \quad ' = \frac{d}{d\theta}\ . \tag{1.22a}$$

Remark that the sign is different from the usual one. We give the bracket through the action (1.17), so ξ means $-\xi(\theta)\dfrac{d}{d\theta}$ in the usual sense.

We identify $\mathcal{N}$ and its Lie algebra $\underline{\mathcal{N}}$: $\underline{\mathcal{N}} = \mathcal{N}$. Since $\mathcal{N}$ is abelian, we have

$$[h_1, h_2] = 0 \quad \text{for} \quad h_1, h_2\ \in\ \underline{\mathcal{N}}\ . \tag{1.22b}$$

We also have, in $\underline{\mathcal{D} \ltimes \mathcal{N}}$

$$[\xi, h] = -\xi h' \quad \text{for} \quad \xi \in \underline{\mathcal{D}}\ ,\ h \in \underline{\mathcal{N}}\ . \tag{1.22c}$$

where the right hand side of (1.22c) is considered as an element of $\underline{\mathcal{N}}$.

In the complexifications $\underline{\mathcal{D}} \otimes \mathbf{C}$ and $\underline{\mathcal{N}} \otimes \mathbf{C}$, we take the bases

$$\ell_n = \ell_n(\theta) = e^{in\theta}\ \in\ \underline{\mathcal{D}} \otimes \mathbf{C}, \quad n \in \mathbf{Z}, \tag{1.23}$$

$$j_n = j_n(\theta) = e^{in\theta}\ \in\ \underline{\mathcal{N}} \otimes \mathbf{C}, \quad n \in \mathbf{Z}. \tag{1.24}$$

We remark that, although the functional forms of $\ell_n(\theta)$ and $j_n(\theta)$ are the same, the meanings as actions on $S^1\mathbf{R}^N$ (more precisely its complexification $(S^1\mathbf{R}^N)^{\mathbf{C}} = S^1\mathbf{C}^N$) are different. Then, from (1.22 a – c), we have

$$[\ell_m, \ell_n] = i(m-n)\,\ell_{m+n}\,, \tag{1.25a}$$

$$[j_m, j_n] = 0\,, \tag{1.25b}$$

$$[\ell_m, j_n] = -in\,j_{m+n}\,. \tag{1.25c}$$

Thus the relations (1.13 a – c) are considered as a central extension of (1.25 a – c) under the correspondence $\hat{L}_n \leftrightarrow \ell_n$, $\hat{a}_n \leftrightarrow j_n$ and

(1.26) $[\ ,\ ]$ in $\underline{\mathcal{D} \ltimes \mathcal{N}}$ $\leftrightarrow$ $i[\ ,\ ]$ in the operator algebra of the Hilbert space

where the Hilbert space is in our case B.

Through Noether's Theorem, the bracket $[\ ,\ ]$ in the Lie algebra of the symmetry group is transformed into the Poisson bracket $\{\ ,\ \}$ of Noether charges with a possibility of cocycle terms ((iii) of Theorem 1 in §2). We also have the correspondence under the canonical quantization:

(1.27) $\{\ ,\ \}$ $\leftrightarrow$ $i[\ ,\ ]$ in the operator algebra of the Hilbert space

with a possibility of the appearance of cocycle terms, in this case by the socalled *anomaly*. These correspondences give the group theoretical meaning of (1.13 a – c).

The above situation is represented by a classical system of free bosons with a symmetry governed by the group $\mathcal{D} \ltimes \mathcal{N}$ (the history is a converse one of course). Noether's theorem yields conserved quantities (Noether charges) whose Poisson brackets represent (with a possible cocycle term) the Lie algebra of the symmetry group.

In this note, we attempt to clarify the following points:

a) On what space and how does the symmetry group $\mathcal{D} \times \mathcal{N}$ act?

b) What is the explicit statement of Noether's theorem which we apply to the situation?

c) What is the meaning of the Sugawara relation (1.10) in the classical level?

In §2, we give an answer to b), Theorem 1. The proof is given in §5, where the proof of a key formula is postponed to the appendix. Theorem 1 is considered as a case which satisfies the assumption of Theorem 4.2.8 in [1]. Namely, after linearization,

$$(1.28) \qquad J \,:\, TM \;\rightarrow\; \mathbf{g}^* \,;\; w \;\mapsto\; \left[\xi \mapsto I_\xi{}^{(t)}(w)\right]$$

is a momentum mapping (for fixed $t \in \mathbf{R}$). Usually, in the literature (for example, Chap. 4 in [1], especially statements above Theorem 4.2.8, Chap. 3 in [7] or Appendix 5 in [2]), the case, where the symmetry group acts on the configuration space M or TM (or T^*M), is considered. So there is a novelty in Theorem 1, where we let the symmetry group act on the space of all paths on M. (In field theory, such a situation is often considered, but it seems that such a relation as (2.27) has not been stated explicitly.)

In §3, we consider the classical system of free bosons with the conformal symmetry corresponding to $\mathcal{D}$. In this case, the cocycle term in (iii) does not appear, hence the cocycle term in the right hand side of (1.13a) is a quantum one.

In §4, the (abelian) gauge symmetry corresponding to $\mathcal{N}$ is considered. In this case, there appears the cocycle term of (iii), thus the cocycle term in the right hand side of (1.13b) is a classical one.

We answer to a) in §3 and §4. An answer to c) is given in §4.

We apply Noether's theorem of finete degrees of freedom in §2 to the system in §3 and §4 which is of infinitely degrees of freedom. So the application is formal.

In §5, we also briefly explain the symplectic geometry, emphasizing the point of view of Lie algebras and homomorphisms between them. It clarifies the reason why, in the relation of the Poisson bracket of Noether charges ((iii) in Theorem 1), the Lie bracket of the Lie algebra of the symmetry group and the cocycle term appear.

We hope the above setting is a starting point to understand the symplectic structure of the Wess–Zumino–Witten model [6]. There are some ambiguities such as: What are the configuration space and the Lagrangian function(al)? What is the meaning of the calculations of Poisson brackets in [6]? To understand such points is our forthcoming problem.

§2 Noether's Theorem

Let M be a configuration space (C^∞ manifold, but we use notations as if manifolds are Euclidian spaces for simplicity) of a classical system of N-degrees of freedom:

$$q = (q^i) = (q^1, q^2, \cdots, q^N) \in M . \tag{2.1}$$

Although this section is written for finite dimensional cases, the results for infinite dimensional cases are also valid.

The system is governed by a Lagrangian function

$$L = L(q, v) : TM \to \mathbf{R} \tag{2.2}$$

where $(q, v) = (q^1, \cdots, q^N; v^1, \cdots, v^N)$ is a point of the tangent bundle TM (we have $v^i = \dot{q}^i = \frac{d}{dt} q^i$ in mind). The motion is determined by the principle of "least" action:

$$\delta \int dt\, L(q(t), \dot{q}(t)) = 0 , \quad \dot{} = \frac{d}{dt} \tag{2.3}$$

whose Euler–Lagrange equation is

$$\frac{d}{dt} \frac{\partial L}{\partial v^i} (q(t), \dot{q}(t)) = \frac{\partial L}{\partial q^i} (q(t), \dot{q}(t)) . \tag{2.4}$$

For simplicity we assume $L(q, v)$ has the form

$$L(q, v) = \frac{1}{2} a_{ij} v^i v^j - U(q) \tag{2.5}$$

where $U : M \to \mathbf{R}$ is a potential and

(2.6)

(a_{ij}) is symmetric, positive definite $N \times N$ matrix, independent of q and v

(repeated upper–lower indexes are summed over $i, j = 1$ to N). Then we have

$$\frac{\partial L}{\partial v^i}(q, v) = a_{ij} v^j \tag{2.7}$$

and the equation of motion (2.4) becomes

$$\ddot{q}^i = -U^i(q) \tag{2.8}$$

where

$$U^i(q) = a^{ij}U_j(q), \quad U_j(q) = \frac{\partial}{\partial q^j}U(q), \quad (a^{ij}) = (a_{ij})^{-1} \,. \tag{2.9}$$

We consider a Lie group G which acts on the space of paths on M:

$$\mathrm{Path}\, M = \mathrm{Map}(\mathbf{R}, M) \ni q(\cdot) \tag{2.10}$$

where $\cdot$ in $q(\cdot)$ is the dummy of $t \in \mathbf{R}$. We consider a situation under which

$$\begin{cases} \text{if } q(\cdot) \in \mathrm{Path}\, M \text{ is a solution of (2.4),} \\ \text{then so is } \gamma \cdot ((q(\cdot)) \text{ for any } \gamma \in G. \end{cases} \tag{2.11}$$

This is the case if the following condition is satisfied (we denote by g the Lie algebra of G):

$$\begin{cases} \text{for } \forall \xi \in \mathrm{g}, \text{ there exists } \Lambda_\xi = \Lambda_\xi(t,q,v) \text{ such that} \\ \text{for } \forall q(\cdot) \in \mathrm{Path}\, M, \text{ putting } q^\epsilon(\cdot) = (\exp - \epsilon\xi)\cdot(q(\cdot)) \text{ , we have} \\ \displaystyle\frac{d}{d\epsilon} L(q^\epsilon(t), \dot{q}^\epsilon(t))\Big|_{\epsilon=0} = \frac{d}{dt}\Lambda_\xi(t, q(t), \dot{q}(t)) \,. \end{cases} \tag{2.12}$$

Under the condition (2.12), if $q(\cdot)$ is a solution of (2.4), then we have

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v^i}(q(t), \dot{q}(t)) \left(\frac{d}{d\epsilon} q^{\epsilon,i}(t)\Big|_{\epsilon=0} \right) - \Lambda_\xi(t, q(t), \dot{q}(t)) \right] = 0 \,. \tag{2.13}$$

In fact, putting

$$A^i(t) = \frac{d}{d\epsilon} q^{\epsilon,i}(t)\Big|_{\epsilon=0} \tag{2.14}$$

and remarking $\displaystyle\frac{d}{d\epsilon}\dot{q}^\epsilon(t)\Big|_{\epsilon=0} = \frac{\partial}{\partial t}\frac{\partial}{\partial \epsilon} q^\epsilon(t)\Big|_{\epsilon=0} = \dot{A}(t)$, we have for solution $q(t)$ of (2.4)

$$
\begin{aligned}
(2.15) \qquad \frac{d}{d\epsilon} L(q^\epsilon(t), \dot{q}^\epsilon(t))\Big|_{\epsilon=0} &= \frac{\partial L}{\partial q^i} A^i + \frac{\partial L}{\partial v^i} \dot{A}^i \\
&= \left(\frac{d}{dt}\frac{\partial L}{\partial v^i}\right) A^i + \frac{\partial L}{\partial v^i} \dot{A}^i \qquad \text{(by (2.4))} \\
&= \frac{d}{dt}\left[\frac{\partial L}{\partial v^i} A^i\right] ,
\end{aligned}
$$

on the other hand this is equal to $\frac{d}{dt}\Lambda_\xi$ by (2.12), giving (2.13). This is the fundamental idea of Noether's Theorem.

To obtain a function $I_\xi = I_\xi(t, q, v) \in C^\infty(\mathbf{R} \times TM)$ for which we have

$$
(2.16) \qquad \text{if } q(\cdot) \text{ is a solution of (2.4), then } \frac{d}{dt} I_\xi(t, q(t), \dot{q}(t)) = 0 ,
$$

we entail the following condition for the G–action on Path M:

$$
(2.17) \quad \begin{cases} \text{for } \forall \xi \in \mathrm{g}, \text{ there exists } A_\xi = A_\xi(t, q, v) \\ \text{such that for } \forall q(\cdot) \in \mathrm{Path}\, M, \text{ we have} \\ \dfrac{d}{d\epsilon} q^\epsilon(t)\Big|_{\epsilon=0} = A_\xi(t, q(t), \dot{q}(t)) , \text{ where } q^\epsilon(\cdot) = (\exp - \epsilon\xi)\cdot(q(\cdot)). \end{cases}
$$

This condition ensures a "locality"of the action. Under the condition (2.17), putting

$$
(2.18) \qquad I_\xi(t, q, v) = \frac{\partial L}{\partial v^i}(q, v) A_\xi^i(t, q, v) - \Lambda_\xi(t, q, v) ,
$$

we have (2.16) by (2.13) (assuming (2.12) of course). This is the first part (i) of Noether's Theorem. We call I_ξ the *Noether charge* (a naming having an application to field theories in mind) for $\xi \in \mathrm{g}$.

To yield the relation on Poisson brackets of Noether charges, we finally entail

$$
(2.19) \qquad A_\xi(t, q, v) \text{ , appeared in (2.17), is linear in } v
$$

(see (A.1) for the exact meaning).

Notation: for a function $f = f(t,q,v)$ of t, q and v, we denote by $f^{(t)}$ the function of q and v given by $f^{(t)}(q,v) = f(t,q,v)$. Under such situations, we have

Theorem 1. (Noether's theorem)

Let M be a configuration space and consider a Lagrangian $L = L(q,v)$ of the form (2.5) with (2.6). We assume a Lie group G acts on Path M so as to satisfy (2.12). We also assume that the G-action satisfies (2.17) and (2.19). Then, for ${I_\xi}^{(t)} \in C^\infty(TM)$, $\xi \in \mathrm{g}$, given by (2.18), the followings are valid.

(i) *If $q = q(t)$ is a solution of (2.4), then we have*

$$\frac{d}{dt} I_\xi(t, q(t), \dot{q}(t)) = 0 \quad \mathit{for}\ \forall \xi \in \mathrm{g} \ .$$

(ii) *The Noether charge ${I_\xi}^{(t)} \in C^\infty(TM)$ generates the transformation given by $\exp - \epsilon\xi$ through the Poisson bracket (see (5.15) or (5.16)) in the following sense:*

$$\text{(2.20a)} \qquad \left\{{I_\xi}^{(t)}, q^i\right\} = {A_\xi}^{(t)i} \ ,$$

$$\text{(2.20b)} \qquad \left\{{I_\xi}^{(t)}, v^i\right\} = {B_\xi}^{(t)i} \ ,$$

where

$$\text{(2.21)} \qquad B_\xi(t,q,v) = \frac{\partial}{\partial t} A_\xi(t,q,v) + \frac{\partial A_\xi}{\partial q^i}(t,q,v)\, v^i - \frac{\partial A_\xi}{\partial v^i}(t,q,v)\, U^i(q) \ .$$

(iii) *For $\forall \xi, \eta \in \mathrm{g}$, we have*

$$\left\{{I_\xi}^{(t)}, {I_\eta}^{(t)}\right\} = I^{(t)}_{[\xi,\eta]} \qquad \text{(up to cocycle)} \ .$$

The proof shall be given in §5.

Remarks:

a) For $f = f(q,v)$, $g = g(q,v) \in C^\infty(TM)$, the Poisson bracket $\{f,g\} \in C^\infty(TM)$ is given by

$$\text{(2.22)} \qquad \{f,g\} = a^{ij}\left(\frac{\partial f}{\partial v^i}\frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial v^j}\right) \ ,$$

where $(a^{ij}) = (a_{ij})^{-1}$ and (a_{ij}) is the one in (2.5). This means we pull-back the canonical Poisson bracket (5.18) for T^*M, the cotangent bundle of M, by the isomorphism

$$(2.23)\qquad \partial_2 L\ :\ TM\ \to\ T^*M\ ;\ (q,v)\ \mapsto\ (q,p),\ p_i = \frac{\partial L}{\partial v^i}(q,v) = a_{ij}v^j\ ,$$

where $\partial_2 L$ means the fibre derivative of L, denoted as FL in Difinition 3.5.2 in [1]. See §5 for a detailed discussion.

b) The definition (2.21) of B_ξ means we have

$$(2.24)\qquad B_\xi(t,q(t),\dot q(t)) = \frac{d}{dt}A_\xi(t,q(t),\dot q(t))$$

for any solution $q(t)$ of (2.4) (or (2.8) for our Lagrangian (2.5)). In the sense we may write as $B_\xi(t,q,v) = \frac{d}{dt}A_\xi(t,q,v)$.

c) The precise meaning of (iii) of the theorem is as follows. In general, an antisymmetric bilinear form $\alpha : \mathbf{g}\times\mathbf{g}\to\mathbf{R}$ is called a *cocycle* if it satisfies

$$(2.25)\qquad \alpha([\xi,\eta],\zeta)+\alpha([\eta,\zeta],\xi)+\alpha([\zeta,\xi],\eta)=0\ ,\quad \text{for } \forall\xi,\eta \text{ and } \zeta\ \in\ \mathbf{g}\ .$$

Then (iii) means that, adding a constant to each Λ_ξ if necessary, the correspondence (for each t)

$$(2.26)\qquad \mathbf{g}\ \to\ C^\infty(TM)\ ;\xi\ \mapsto\ I_\xi{}^{(t)}$$

is linear and there exists a cocycle $\alpha^{(t)} : \mathbf{g}\times\mathbf{g}\to\mathbf{R}$ for which we have

$$(2.27)\qquad \left\{I_\xi{}^{(t)}, I_\eta{}^{(t)}\right\} = I^{(t)}_{[\xi,\eta]} + \alpha^{(t)}(\xi,\eta)\ ,\qquad \text{for}\quad \xi,\eta\ \in\ \mathbf{g}\ .$$

§3 Conformal symmetry of free bosons

We consider a system of free bosons (bosonic closed string model) described by the Lagrangian density (Chap. 2 in [3])

$$(3.1)\qquad \mathcal{L}(X) = \frac{1}{2}\partial^\mu X\partial_\mu X = \frac{1}{2}\left\{(\partial_0 X)^2-(\partial_1 X)^2\right\}\ ,$$

where $x=(x^\mu)=(x^0,x^1)\ \in\ \mathbf{R}^2$, $X=X(x)\ \in\ \mathbf{R}^N$ and $\partial_\mu=\partial/\partial x^\mu$, $\partial^\mu=\eta^{\mu\nu}\partial_\nu$, $(\eta_{\mu\nu})=\mathrm{diag}[1,-1]$, $(\eta^{\mu\nu})=(\eta_{\mu\nu})^{-1}$. We also take $\tau=x^0$ and $\sigma=x^1$, then (3.1) becomes

$$(3.2)\qquad \mathcal{L}(X)=\frac{1}{2}\{\dot X^2-X'^2\}\ ,\quad \dot{}=\frac{\partial}{\partial\tau}\ ,\quad {}'=\frac{\partial}{\partial\sigma}\ .$$

The action functional is given by

$$S[X] = \int d\tau d\sigma\, \mathcal{L}(X) \tag{3.3}$$

and the equation of motion (Euler–Lagrange equation of $\delta S[X] = 0$) is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu X} = \frac{\partial \mathcal{L}}{\partial X} . \tag{3.4}$$

For our case we have $\partial \mathcal{L} / \partial \partial_\mu X = \partial^\mu X$, hence (3.4) becomes

$$\partial_\mu \partial^\mu X = 0 \qquad \text{i.e.,} \quad \partial_0 \partial_0 X - \partial_1 \partial_1 X = 0 . \tag{3.5}$$

We consider the closed string case: $X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$, and regarding τ as time (written as t in §2), we take the configuration space

$$M = S^1 \mathbf{R}^N = \{\ X = X(\sigma) \ \in \ \mathbf{R}^N \text{ with } X(\sigma + 2\pi) = X(\sigma)\ \} . \tag{3.6}$$

Since $S^1 \mathbf{R}^N$ is a linear space, the tangent bundle is

$$TM = S^1 \mathbf{R}^N \times S^1 \mathbf{R}^N \quad \ni \quad (X(\Box), Y(\Box)) , \tag{3.7}$$

where $\Box$ is the dummy of σ and $Y(\sigma) = \dot{X}(\sigma)$ in mind. The Lagrangian L is given by

$$L \ : \ TM \ \to \ \mathbf{R}\ ; \ (X(\Box), Y(\Box)) \ \mapsto \ \frac{1}{2} \int_0^{2\pi} d\sigma \left\{ Y(\sigma)^2 - X'(\sigma)^2 \right\} \tag{3.8}$$

giving $S[X]$ of (3.3) after further τ integration.

To give the conformal (and later gauge) symmetry, we take the light–cone coordinate:

$$x^+ = x^0 + x^1 , \quad x^- = x^0 - x^1 \tag{3.9}$$

and consider an orientation preserving diffeo of S^1 (considered as a space of x^+) :

$$\phi : S^1 \to S^1 \text{ (or } \mathbf{R} \to \mathbf{R} \text{ with } \phi(x^+ + 2\pi) = \phi(x^+) + 2\pi) . \tag{3.10}$$

Then the diffeo given by

$$\tilde{\phi} \ : \mathbf{R} \times S^1 \ \to \ \mathbf{R} \times S^1 \ ; \ (x^+, x^-) \ \mapsto \ (\phi(x^+), x^-) , \tag{3.11}$$

or returning to the original coordinates

$$\text{(3.12)} \quad \tilde{\phi} \,:\, (x^0, x^1) \mapsto \left(\tfrac{1}{2}\{\phi(x^0+x^1) + x^0 - x^1\}, \tfrac{1}{2}\{\phi(x^0+x^1) - x^0 + x^1\}\right)$$

is a conformal mapping with respect to $(\eta_{\mu\nu})$, that is

$$\text{(3.13)} \qquad (\tilde{\phi}^*\eta)_x = \Omega(x)\eta \quad \text{for some function } \Omega = \Omega(x) > 0 \;,$$

where $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$ is the Lorenztian metric on $\mathbf{R} \times S^1$.

Considering infinitesimally, we take

$$\text{(3.14)} \qquad \xi \,:\, S^1 \to \mathbf{R} \quad (\text{or } \mathbf{R} \to \mathbf{R} \text{ with } \xi(x^+ + 2\pi) = \xi(x^+))$$

and for $\epsilon \in \mathbf{R}$, $|\epsilon| \ll 1$,

$$\text{(3.15)} \qquad \phi(x^+) = x^+ + \epsilon\xi(x^+) \;.$$

(Hereafter we omit the higher order terms of ϵ for simplicity.) Then $\tilde{\phi}$ written with the coordinate (τ, σ) is

$$\text{(3.16)} \qquad (\tau,\sigma) \mapsto (\tau + \tfrac{1}{2}\epsilon\xi(\tau+\sigma), \sigma + \tfrac{1}{2}\epsilon\xi(\tau+\sigma)) \;.$$

Through this conformal transformation of $\tau\sigma$–space, we consider the transformation of field $X(\tau,\sigma)$:

$$\text{(3.17)} \qquad X(\tau,\sigma) \mapsto X^\epsilon(\tau,\sigma) = X(\tau + \tfrac{1}{2}\epsilon\xi(\tau+\sigma), \sigma + \tfrac{1}{2}\epsilon\xi(\tau+\sigma)) \;.$$

or in the light–cone coordinate

$$\text{(3.18)} \qquad X(x^+, x^-) \mapsto X^\epsilon(x^+, x^-) = X(x^+ + \epsilon\xi(x^+), x^-) \;.$$

The correspondence to the notations in §2 is as follows. As the symmetry group G, we take $\mathcal{D}$ of (1.14) with Lie algebra $\underline{\mathcal{D}}$ ($\theta \leftrightarrow x^+$ in mind). The path space is in this case

$$\text{(3.19)} \quad \operatorname{Path} M = \operatorname{Map}(\mathbf{R}, \operatorname{Map}(S^1, \mathbf{R}^N)) = \operatorname{Map}(\mathbf{R} \times S^1, \mathbf{R}^N) \ni X(\cdot, \square) \;.$$

The action of $\mathcal{D}$ on $\operatorname{Path} M$ is

(3.20)

$$(\phi \cdot X)(\tau,\sigma) = X(\tilde{\phi}^{-1}(\tau,\sigma)) \;, \quad \text{for} \quad \phi \in \mathcal{D},\; X = X(\tau,\sigma) \in \operatorname{Path} M \;,$$

hence, infinitesimally for $\xi \in \underline{\mathcal{D}}$, if we put

$$X^{\epsilon}(\cdot,\square) = (\exp - \epsilon\xi)\cdot(X(\cdot,\square)) \ , \tag{3.21}$$

we have

$$X^{\epsilon}(\tau,\sigma) = X(\tau + \tfrac{1}{2}\epsilon\xi(\tau+\sigma), \sigma + \tfrac{1}{2}\epsilon\xi(\tau+\sigma)) \ , \tag{3.22}$$

giving (3.17).

For this $\mathcal{D}$–action, we can show the existence of Λ_ξ of (2.12) as follows. Remark that, in light–cone coordinates, we have

$$\mathcal{L}(X) = 2\partial_+ X \partial_- X \ . \tag{3.23}$$

For $\xi \in \underline{\mathcal{D}}$, taking $X^\epsilon = X^\epsilon(\tau,\sigma)$ as (3.22) or

$$X^\epsilon(x^+,x^-) = X(x^+ + \epsilon\xi(x^+), x^-) = X(x^+,x^-) + \epsilon\xi(x^+)\partial_+ X(x^+,x^-) \ , \tag{3.24}$$

we have

$$\begin{aligned} \frac{d}{d\epsilon}\mathcal{L}(X^\epsilon)\Big|_{\epsilon=0} &= 2\left(\frac{d}{d\epsilon}\partial_+ X^\epsilon\Big|_{\epsilon=0}\right)\partial_- X + 2\partial_+ X\left(\frac{d}{d\epsilon}\partial_- X^\epsilon\Big|_{\epsilon=0}\right) \\ &= 2\{\xi'(x^+)\partial_+ X + \xi(x^+)\partial_+\partial_+ X\}\partial_- X \\ &\qquad + 2\partial_+ X \xi(x^+)\partial_-\partial_+ X \\ &= 2\partial_+\{\xi(x^+)\partial_+ X \partial_- X\} \\ &= \frac{1}{2}(\partial_0 + \partial_1)[\xi(\tau+\sigma)\mathcal{L}] \\ &= \partial_\mu\left[\frac{1}{2}(\delta_0^\mu + \delta_1^\mu)\xi(\tau+\sigma)\mathcal{L}\right] \ , \end{aligned} \tag{3.25}$$

where δ_ν^μ is Kronecker's delta. Thus taking the σ–integral for $\mu = 0$ component of [ ] of the final formula,

$$\Lambda_\xi(\tau, X(\square), Y(\square)) = \frac{1}{2}\int_0^{2\pi} d\sigma\, \xi(\tau+\sigma)\cdot\frac{1}{2}\{Y(\sigma)^2 - X'(\sigma)^2\} \tag{3.26}$$

gives the desired function(al):

$$\frac{d}{d\epsilon}L(X^\epsilon(\tau,\square), \dot{X}^\epsilon(\tau,\square))\Big|_{\epsilon=0} = \frac{d}{d\tau}\Lambda_\xi(\tau, X(\tau,\square), \dot{X}(\tau,\square)) \ . \tag{3.27}$$

Therefore all assumptions of Noether's theorem are satisfied (for the linearity (2.19) in v, in our case $Y = \partial_0 X$, see (3.31)).

The corresponding Noether charge

$$I_\xi = I_\xi(\tau, X(\square), Y(\square)) \in C^\infty(\mathbf{R} \times TM) \tag{3.28}$$

is given as follows. To obtain the first term of the right hand side of (2.18):

$$\frac{\partial L}{\partial v^i} A^i_\xi(t, q, v) \ , \tag{3.29}$$

we remark that

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu X} = \partial^\mu X \tag{3.30}$$

and

$$\left.\frac{d}{d\epsilon} X^\epsilon\right|_{\epsilon=0} = \xi(x^+)\partial_+ X = \frac{1}{2}\xi(\tau+\sigma)(\partial_0 X + \partial_1 X) \ . \tag{3.31}$$

Thus (3.29) corresponds to the σ–integral of the $\mu = 0$ component of

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu X}\left(\left.\frac{d}{d\epsilon} X^\epsilon\right|_{\epsilon=0}\right) = \frac{1}{2}\,\partial^\mu X\,[\xi(\tau+\sigma)(\partial_0 X + \partial_1 X)] \ . \tag{3.32}$$

Hence we have

$$\begin{aligned} &I_\xi(\tau, X(\square), Y(\square)) \\ &= \frac{1}{2}\int_0^{2\pi} d\sigma\, \xi(\tau+\sigma)\left[Y(\sigma)(Y(\sigma)+X'(\sigma)) - \tfrac{1}{2}(Y(\sigma)^2 - X'(\sigma)^2)\right] \\ &= \frac{1}{4}\int_0^{2\pi} d\sigma\, \xi(\tau+\sigma)\,[Y(\sigma)+X'(\sigma)]^2 \ . \end{aligned} \tag{3.33}$$

For these Noether charges ${I_\xi}^{(\tau)}$, one can verify (i) and (ii) of Noether's Theorem directly.

For (iii), Noether's Theorem does not determine the cocycle term $\alpha^{(t)}(\xi, \eta)$ in (2.27). So we compute the Poisson brackets actually. In this case, we have $\alpha^{(t)} = 0$ as follows.

For $\xi \in \underline{\mathcal{D}}$, we have

$$\text{(3.34a)} \qquad \frac{\delta I_\xi{}^{(\tau)}}{\delta X(\sigma)} = -\frac{1}{2}[\xi'(\tau+\sigma)(Y(\sigma)+X'(\sigma)) + \xi(\tau+\sigma)(Y'(\sigma)+X''(\sigma))] \ ,$$

$$\text{(3.34b)} \qquad \frac{\delta I_\xi{}^{(\tau)}}{\delta Y(\sigma)} = \frac{1}{2}\xi(\tau+\sigma)[Y(\sigma)+X'(\sigma)] \ .$$

Thus for $\xi,\ \eta\ \in\ \underline{\mathcal{D}}$, we have

$$\text{(3.35)} \qquad \{I_\xi{}^{(\tau)}, I_\eta{}^{(\tau)}\}(X(\Box), Y(\Box))$$

$$= \int_0^{2\pi} d\sigma \left[\frac{\delta I_\xi{}^{(\tau)}}{\delta Y(\sigma)}\frac{\delta I_\eta{}^{(\tau)}}{\delta X(\sigma)} - \frac{\delta I_\xi{}^{(\tau)}}{\delta X(\sigma)}\frac{\delta I_\eta{}^{(\tau)}}{\delta Y(\sigma)}\right]$$

$$= \frac{1}{4}\int_0^{2\pi} d\sigma\ [\xi'(\tau+\sigma)\eta(\tau+\sigma) - \xi(\tau+\sigma)\eta'(\tau+\sigma)]\,[Y(\sigma)+X'(\sigma)]^2$$

$$= I^{(\tau)}_{[\xi,\eta]}(X(\Box), Y(\Box)) \ ,$$

here we recall (1.22a). This means $\alpha^{(t)} = 0$. The above normalization of Poisson bracket $\{\ \ ,\ \ \}$ on $C^\infty(TM)$ is given as follows. If we give $S^1\mathbf{R}^N$ the "inner product" $<Y_1, Y_2>$ by $\int_0^{2\pi} d\sigma\, Y_1(\sigma)\, Y_2(\sigma)$, then the form of the Lagrangian (3.8) means the finite dimensional corresponding a_{ij} in (2.5) is Kronecker's delta. So (2.22) gives the normalization.

Using the energy–momentum tensor

$$\text{(3.36)} \qquad T^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu X}\partial_\nu X - \delta^\mu_\nu\mathcal{L} = \partial^\mu X\partial_\nu X - \delta^\mu_\nu\mathcal{L} \ ,$$

the Noether charge $I_\xi{}^{(\tau)}$ is written as follows. From (3.25) and (3.32), we see that the Noether charge $I_\xi{}^{(\tau)}$ is the σ–integral of the $\mu = 0$ component of

(3.37)

$$\frac{1}{2}\xi(\tau+\sigma)\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu X}(\partial_0 X + \partial_1 X) - (\delta^\mu_0 + \delta^\mu_1)\mathcal{L}\right] = \frac{1}{2}\xi(\tau+\sigma)\,[T^\mu{}_0 + T^\mu{}_1] \ .$$

In the light–cone coordinate, we have (see (2.1.40) in [3])

$$\text{(3.38)} \qquad T_{++} = \frac{1}{2}(T_{00}+T_{01}) = (\partial_+ X)^2 = \frac{1}{4}(\partial_0 X + \partial_1 X)^2$$

where

$$T_{\mu\nu} \equiv \eta_{\mu\rho}T^{\rho}{}_{\nu} = \partial_\mu\partial_\nu X - \eta_{\mu\nu}\mathcal{L} \ . \tag{3.39}$$

Thus we have

$$I_\xi{}^{(\tau)}(X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\, \xi(\tau+\sigma)T_{++} \ , \tag{3.40}$$

where, in the right hand side, $\partial_0 X$ in T_{++} is replaced by $Y(\sigma)$.

So far we consider the real Lie algebra $\underline{\mathcal{D}} \ni \xi$ and the real valued functions $C^\infty(TM) \ni I_\xi{}^{(\tau)}$. For fixed $\tau \in \mathbf{R}$, we extend the correspondence $\xi \mapsto I_\xi{}^{(\tau)}$ to the complexified one:

$$\underline{\mathcal{D}} \otimes \mathbf{C} \to C^\infty(TM)^{\mathbf{C}} \equiv \mathrm{Map}(TM, \mathbf{C})$$

by the obvious fashion. We define $L_n{}^{(\tau)} \in C^\infty(TM)^{\mathbf{C}}$, $n \in \mathbf{Z}$, by

$$L_n{}^{(\tau)}(X(\square), Y(\square)) \equiv I_{\ell_n}{}^{(\tau)}(X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\, e^{in(\tau+\sigma)}T_{++} \ , \tag{3.41}$$

where $\ell_n \in \underline{\mathcal{D}}\otimes\mathbf{C}$ is the one in (1.23) with $\theta \leftrightarrow x^+$. Then we have, from (3.35)

$$\begin{aligned} \{L_m{}^{(\tau)}, L_n{}^{(\tau)}\}(X(\square), Y(\square)) &= i(m-n)\int_0^{2\pi} d\sigma\, e^{i(m+n)(\tau+\sigma)}T_{++} \\ &= i(m-n)L_{m+n}^{(\tau)}(X(\square), Y(\square)) \ . \end{aligned} \tag{3.42}$$

This is the relation in the classical level corresponding to (1.13a) without cocycle term.

§4 Gauge symmetry of free bosons and Sugawara relation

The system of free bosons, whose configuration space is (3.6), with the Lagrangian (3.8) has also the following (abelian) gauge symmetry.

Let the group $\mathcal{N}$ of (1.15) act on Path M (3.19) as (recall the correspondence $\theta \leftrightarrow x^+$)

$$X^\epsilon(x^+, x^-) = X(x^+, x^-) + \epsilon h(x^+) \ , \qquad \text{for } h \in \mathcal{N} \tag{4.1}$$

or

(4.2) $$X^\epsilon(\tau,\sigma) = X(\tau,\sigma) + \epsilon h(\tau+\sigma)\ , \qquad \text{for } h \in \mathcal{N}$$

here we have written the final form corresponding to (3.22) for the case of the conformal symmetry. (To obtain this, the action (1.18) must be $(h\cdot X)(\theta) = X(\theta) - h(\theta)$. This changes j_n to $-j_n$, retaining the relation (1.25 a – c).) In this case, we have, using (3.23),

(4.3) $$\begin{aligned} \frac{d}{d\epsilon}\mathcal{L}(X^\epsilon)\bigg|_{\epsilon=0} &= 2\left(\frac{d}{d\epsilon}\partial_+ X^\epsilon\bigg|_{\epsilon=0}\right)\partial_- X + 2\partial_+ X\left(\frac{d}{d\epsilon}\partial_- X^\epsilon\bigg|_{\epsilon=0}\right) \\ &= 2\,h'(x^+)\,\partial_- X \\ &= h'(\tau+\sigma)\,(\partial_0 X - \partial_1 X) \\ &= \partial_0\{-h(\tau+\sigma)\,\partial_1 X\} + \partial_1\{h(\tau+\sigma)\,\partial_0 X\} \\ &= \partial_\mu\,[-\epsilon^{\mu\nu}\,h(\tau+\sigma)\,\partial_\nu X]\ , \end{aligned}$$

here $\epsilon^{\mu\nu}$ is the antisymmetric tensor with $\epsilon^{01}=1$. Thus $\Lambda_h(\tau, X(\square), Y(\square))$ is obtained as the σ–integral of the $\mu=0$ component of [ ] of the final formula of (4.3):

(4.4) $$\Lambda_h(\tau, X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\ [-h(\tau+\sigma)\,X'(\sigma)]\ .$$

Hence all assumptions of Theorem 1 are satisfied.

The Noether charge $I_h(\tau, X(\square), (\square))$ is the σ–integral of the $\mu=0$ component of

(4.5) $$\frac{\partial\mathcal{L}}{\partial\partial_\mu X}\left(\frac{d}{d\epsilon}X^\epsilon\bigg|_{\epsilon=0}\right) - [-\epsilon^{\mu\nu}\,h(\tau+\sigma)\,\partial_\nu X] = h(\tau+\sigma)\ [\partial^\mu X + \epsilon^{\mu\nu}\partial_\nu X]\ ,$$

that is,

(4.6) $$I_h(\tau, X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\, h(\tau+\sigma)\,[Y(\sigma) + X'(\sigma)]\ .$$

For this Noether charge, we have

(4.7a) $$\frac{\delta I_h^{(\tau)}}{\delta X(\sigma)} = -h'(\tau+\sigma)\ ,$$

$$\frac{\delta I_h^{(\tau)}}{\delta Y(\sigma)} = h(\tau+\sigma) \ . \tag{4.7b}$$

Hence, for $h_1, h_2 \in \mathcal{N}$, we have

$$\begin{aligned} \left\{I_{h_1}^{(\tau)}, I_{h_2}^{(\tau)}\right\} &= \int_0^{2\pi} d\sigma\, [h_1'(\tau+\sigma)\, h_2(\tau+\sigma) - h_1(\tau+\sigma)\, h_2'(\tau+\sigma)] \\ &= \int_0^{2\pi} d\sigma\, [h_1'(\sigma)\, h_2(\sigma) - h_1(\sigma)\, h_2'(\sigma)] \ . \end{aligned} \tag{4.8}$$

We remark that the right hand side is a constant as a function of $(X(\square), Y(\square))$ and defines a cocycle (since $\mathcal{N}$ is abelian, only the anti–symmetricity is required).

After complexification, we define $J_n^{(\tau)} \in C^\infty(TM)^{\mathbf{C}}$ as $I_h^{(\tau)}$ for $h = h(\theta) = e^{in\theta}/\sqrt{4\pi} \in \underline{\mathcal{N}} \otimes \mathbf{C}$:

$$J_n^{(\tau)}(X(\square), Y(\square)) = \frac{1}{\sqrt{4\pi}} \int_0^{2\pi} d\sigma\, e^{in(\tau+\sigma)}\, [Y(\sigma) + X'(\sigma)] \ . \tag{4.9}$$

Then we have from (4.8)

$$\left\{J_m^{(\tau)}, J_n^{(\tau)}\right\} = \frac{1}{4\pi} \int_0^{2\pi} d\sigma\, i(m-n) e^{i(m+n)\sigma} = im\delta_{m+n,0} \ . \tag{4.10}$$

Thus the correspondence

$$J_n^{(\tau)} \longleftrightarrow \hat{a}_n \tag{4.11}$$

with (1.27) gives the relation (1.13b).

To see the correspondence to the Sugawara relation (1.10), we reverse the relation of $\partial_+ X = \frac{1}{2}[Y(\sigma) + X'(\sigma)]$ and $J_n^{(\tau)}$ in (4.9) as follows. First we expand $\partial_+ X$ as

$$\frac{1}{2}[Y(\sigma) + X'(\sigma)] = \sum_{n \in \mathbf{Z}} c_n^{(\tau)}(X(\square), Y(\square))\, e^{in(\tau+\sigma)} \ , \tag{4.12}$$

here $c_n^{(\tau)}(X(\square), Y(\square))$ is the Fourier coefficient showing the dependence on $\tau, X(\square), Y(\square)$ explicitly, and substitute (4.12) to (4.9), to obtain

$$J_n^{(\tau)} = 2\sqrt{\pi} c_{-n}^{(\tau)} \ . \tag{4.13}$$

Thus we have

$$(4.14)\qquad \partial_+ X(\sigma) = \frac{1}{2\sqrt{\pi}} \sum_{n\in\mathbf{Z}} e^{-in(\tau+\sigma)} J_n{}^{(\tau)} ,$$

where the left hand side is considered as the function(al)

$$(4.15)\qquad (X(\square), Y(\square)) \in TM = S^1\mathbf{R}^N \times S^1\mathbf{R}^N \mapsto \frac{1}{2}[Y(\sigma) + X'(\sigma)] .$$

After (1.12), considering the correspondence (4.11), we put

$$(4.16)\qquad J^{(\tau)}(\sigma) = \sum_{n\in\mathbf{Z}} e^{-in(\tau+\sigma)} J_n{}^{(\tau)} ,$$

which is a function(al) on TM. Then (4.14) becomes

$$(4.17)\qquad \partial_+ X(\sigma) = \frac{1}{2\sqrt{\pi}} J^{(\tau)}(\sigma) .$$

Both sides of (4.17) are considered as function(al) on TM.

We apply the same method to (3.41) to obtain

$$(4.18)\qquad T_{++}(\sigma) = [\partial_+ X(\sigma)]^2 = \frac{1}{2\pi} T^{(\tau)}(\sigma)$$

where, after (1.11), we put

$$(4.19)\qquad T^{(\tau)}(\sigma) = \sum_{n\in\mathbf{Z}} e^{-in(\tau+\sigma)} L_n{}^{(\tau)} .$$

In (4.18), $T_{++}(\sigma)$ means the function(al)

$$(4.20)\qquad (X(\square), Y(\square)) \in TM \mapsto \frac{1}{4}[Y(\sigma) + X'(\sigma)]^2 .$$

Thus, comparing (4.17) and (4.18), we have

$$(4.21)\qquad T^{(\tau)}(\sigma) = \frac{1}{2}\left[J^{(\tau)}(\sigma)\right]^2 .$$

This is the clas ' ' relation corresponding to the Sugawara relation (1.10).

Remark that the equality (1.10) is the one as operators on the Hilbert space, but the equality (4.21) is the one as function(al)s on $TM = S^1\mathbf{R}^N \times S^1\mathbf{R}^N$.

§5 Symplectic geometry and proof of Noether's Theorem

To see a general aspect of Noether's theorem, we first point out that we have three Lie algebras. First the Lie algebra g of the Lie group G, the second is the space of all real valued C^∞ functions $C^\infty(TM)$ with the Poisson bracket (2.22), and as the third, the space of vector fields on a manifold with the usual bracket [,] of vector fields.

There are some cases in which we have a homomorphism of these Lie algebras. For example we assume that the Lie group G acts on a manifold Z. For $\xi \in \mathrm{g}$, we define the vector field $V_\xi \in V(Z)$ by (we denote by $V(Z)$ the space of all vector field on Z)

$$(5.1) \qquad V_\xi(z) = \frac{d}{d\epsilon}\left[(\exp - \epsilon\xi)\cdot z\right]\Bigg|_{\epsilon=0} \qquad \text{for} \quad z \in Z \,.$$

Then it is known that (Prop. 4.1.26 in [1]. Remark that our sign convention is different. Ours are chosen so as to make mappings between Lie algebras homomorphism.)

Proposition 5.1 *The correspondence*

$$(5.2) \qquad \xi \in \mathrm{g} \quad \mapsto \quad V_\xi \in V(Z)$$

is a Lie algebra homomorphism, that is, it is linear and

$$(5.3) \qquad [V_\xi, V_\eta] = V_{[\xi,\eta]} \qquad \textit{in} \quad V(Z) \quad \textit{for} \quad \forall \xi, \eta \in \mathrm{g} \,.$$

As a next example, we consider a symplectic manifold W. This means W is a C^∞ manifold with a 2–form ω satisfying

(5.4a) ω is closed, i.e., $d\omega = 0$,

(5.4b) ω is nondegenerate ,

in the sense that for $V \in V(W)$

$$(5.5) \qquad \omega(V, \cdot) = 0 \qquad \text{if and only if} \qquad V = 0 \,,$$

where $\omega(V, \cdot)$, which is also written as $i(V) \cdot \omega$, means the 1–form

$$V_1 \in V(W) \quad \mapsto \quad \omega(V, V_1)\ . \tag{5.6}$$

The 2–form ω is called a *symplectic 2–form.* The condition (5.5) is equivalent to

(5.7) for $\forall$ 1–form α on W, $\exists$ the unique $V \in V(W)$, for which $\alpha = \omega(V, \cdot)$.

For any C^∞ manifold M, its cotangent bundle

$$T^*M \ni (q, p) = (q^1, q^2, \cdots, q^N; p_1, p_2, \cdots, p_N) \tag{5.8}$$

has the canonical symplectic 2–form

$$\omega = dp_i \wedge dq^i \qquad \text{(summed over } i = 1 \text{ to } N)\ . \tag{5.9}$$

We have

$$\omega = d\theta \tag{5.10}$$

where θ is the 1–form on T^*M

$$\theta = p_i \wedge dq^i\ , \tag{5.11}$$

called the *canonical 1–form.* This ω is exact, although we require only closedness for general cases.

Returning to a general symplectic manifold (W, ω), we define, for $f \in C^\infty(W)$, the vector field $V_f \in V(W)$ by

$$-df = \omega(V_f, \cdot) \qquad \text{as 1–forms on } W\ . \tag{5.12}$$

This V_f is uniquely determined from f by virtue of (5.7). It is seen that the correspondence

$$f \in C^\infty(W) \quad \mapsto \quad V_f \in V(W) \tag{5.13}$$

is a Lie algebra homomorphism :

$$[V_f, V_g] = V_{\{f,g\}} \qquad \text{for} \quad f, g \in C^\infty(W)\ , \tag{5.14}$$

where $\{f, g\} \in C^\infty(W)$ is the Poisson bracket defined by

$$\{f, g\} = \omega(V_f, V_g)\ . \tag{5.15}$$

(see Coro. 3.3.18 of [1], remarking the sign convention.) Remark that the definition (5.15) is also written as

$$\{f,g\} = V_f \cdot g \; . \tag{5.16}$$

For a cotangent bundle $W = T^*M$ with the symplectic structure (5.9), we have

$$V_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \, , \qquad \text{for} \quad f = f(q,p) \in C^\infty(T^*M) \, , \tag{5.17}$$

hence, by (5.16)

$$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} \, . \tag{5.18}$$

In general, a vector field $V \in V(W)$ is called *Hamiltonian vector field* if $V = V_f$ for some $f \in C^\infty(W)$. A vector field $V \in V(W)$ is called *locally Hamiltonian vector field* if for any $w \in W$, we can find a C^∞ function f defined on a neighborhood of w for which $-df = \omega(V,\cdot)$ on the neighborhood. We denote by $HV(W)$ (resp. $LHV(W)$) the space of all Hamiltonian (resp. locally Hamiltonian) vector fields on W. In the words of forms, we can say that for $V \in V(W)$,

$$V \in HV(W) \iff \omega(V,\cdot) \; : \; \text{exact} \, , \tag{5.19}$$

$$V \in LHV(W) \iff \omega(V,\cdot) \; : \; \text{closed} \, . \tag{5.20}$$

It is seen that

$$V \in LHV(W) \iff \mathcal{L}_V\omega = 0 \, , \tag{5.21}$$

where $\mathcal{L}_V\omega$ is the Lie derivative (see Chap. 2 in [1], especially p.121), so locally Hamiltonian vector fields are also called *symplectic vector fields.* It is easy to see that $HV(W)$ and $LHV(W)$ are closed under [,], hence we have three Lie algebras:

$$HV(W) \subset LHV(W) \subset V(W) \, .$$

We have from (5.19),(5.20) and (5.7)

$$LHV(W)/HV(W) \cong \mathrm{H}^1(W,\mathbf{R}) \, , \text{de Rham cohomology} \, . \tag{5.22}$$

(Proof of Theorem 1)

(i) is already given in §2.

Let M be the configuration space in the theorem. We pull-back the symplectic structure (5.9) on T^*M to the one on TM by (2.23) :

$$\omega = d\theta = dv_i \wedge dq^i = a_{ij}\, dv^i \wedge dq^j \ , \tag{5.23}$$

$$\theta = v_i dq^i = a_{ij} v^i dq^j \ , \tag{5.24}$$

where we use the same notation ω and θ on TM for simplicity and use (a_{ij}), which has the inverse $(a_{ij})^{-1} = (a^{ij})$ by (2.6), to let the indices $i, j, \cdots$ up or down:

$$v_i = a_{ij} v^j \ , \quad A_{\xi\, i}^{(t)}(q,v) = a_{ij} A_\xi^{(t)\, j}(q,v) \ , \ \cdots, \ \text{etc.} \tag{5.25}$$

Under the assumption of the theorem, for any fixed $t_0 \in \mathbf{R}$, we can obtain a G–action on TM as follows. For $\gamma \in G$ and $(q_0, v_0) \in TM$, let $q = q(t) \in$ Path M be the solution of (2.4) with

$$q(t_0) = q_0, \quad \dot{q}(t_0) = v_0 \tag{5.26}$$

and put

$$q^\gamma(\cdot) = \gamma \cdot (q(\cdot)) \tag{5.27}$$

(we assume all solutions of (2.4) with initial condition as (5.26) can be defined for all $t \in \mathbf{R}$).

Then defining as

$$\gamma \cdot (q_0, v_0) = (q^\gamma(t_0), \dot{q}^\gamma(t_0)) \ , \tag{5.28}$$

we have

Proposition 5.2 *For any fixed $t_0 \in \mathbf{R}$, the above construction (5.28) gives a G–action on TM.*

(*proof*) For given $(q_0, v_0) \in TM$, let $q_1(\cdot)$ be the solution of (2.4) with $q_1(t_0) = q_0, \dot{q}_1(t_0) = v_0$. Then for γ_1 and $\gamma_2 \in G$, $q_2(\cdot) \equiv \gamma_1 \cdot (q_1(\cdot))$ and $q_3(\cdot) \equiv \gamma_2 \cdot (q_2(\cdot)) = (\gamma_2\gamma_1) \cdot (q_1(\cdot))$ are also solutions of (2.4) by (2.11). Here we use the fact that G acts on Path M. Then $\gamma_1 \cdot (q_0, v_0) = (q_2(t_0), \dot{q}_2(t_0))$, $\gamma_2 \cdot (q_2(t_0), \dot{q}_2(t_0)) = (q_3(t_0), \dot{q}_3(t_0)) = (\gamma_2\gamma_1) \cdot (q_0, v_0)$, hence

$$\gamma_2 \cdot (\gamma_1(q_0, v_0)) = (\gamma_2\gamma_1) \cdot (q_0, v_0) \ .$$

This means (5.28) gives a group action. Q.E.D.

For fixed $t \in \mathbf{R}$, using A_ξ in (2.17) and B_ξ in (2.21), we define the vector field on TM by

$$V_\xi^{(t)} = A_\xi^{(t)i}(q,v)\frac{\partial}{\partial q^i} + B_\xi^{(t)i}(q,v)\frac{\partial}{\partial v^i}\ . \tag{5.29}$$

Then the definition of $A_\xi^{(t)}$ and $B_\xi^{(t)}$ just means that the vector field $V_\xi^{(t)}$ is the one given by (5.1) for the G–action (5.28) on TM for $t(=t_0)$. See remark b) after Theorem 1. Thus we have by Proposition 5.1

$$\left[V_\xi^{(t)}, V_\eta^{(t)}\right] = V_{[\xi,\eta]}^{(t)}\ , \qquad \text{for} \quad \xi,\eta \in \mathbf{g}\ , \quad t \in \mathbf{R}\ . \tag{5.30}$$

The key step to prove the theorem is the following

Proposition 5.3 *For $t \in \mathbf{R}$ and $\xi \in \mathrm{g}$, let $V_\xi^{(t)} \in V(TM)$ given by (5.29) and $\Lambda_\xi^{(t)} \in C^\infty(TM)$ be the function in (2.12). Then we have*

$$\mathcal{L}_{V_\xi^{(t)}}\theta = d\Lambda_\xi^{(t)}\ . \tag{5.31}$$

The proof shall be given in the Appendix. From (5.31), we see immediately

$$\mathcal{L}_{V_\xi^{(t)}}\omega = 0\ , \quad \text{i.e.,} \quad V_\xi^{(t)} \in LHV(TM)\ , \tag{5.32}$$

because $\mathcal{L}_{V_\xi^{(t)}}\omega = \mathcal{L}_{V_\xi^{(t)}}(d\theta) = d\mathcal{L}_{V_\xi^{(t)}}\theta = dd\Lambda_\xi^{(t)} = 0$.

We point out here that $I_\xi^{(t)} \in C^\infty(TM)$ of (2.18) is written as

$$I_\xi^{(t)} = \left\langle\theta, V_\xi^{(t)}\right\rangle - \Lambda_\xi^{(t)}\ , \tag{5.33}$$

because of $\left\langle\theta, V_\xi^{(t)}\right\rangle = \left\langle v_i dq^i, A_\xi^{(t)}\frac{\partial}{\partial q} + B_\xi^{(t)}\frac{\partial}{\partial v}\right\rangle = v_i A_\xi^{(t)i}$ and (2.7). Compare this to the formula in Theorem 4.2.10 in [1]. We assume that M is connected but do not assume the vanishing of $\mathrm{H}^1(M,\mathbf{R}) \cong \mathrm{H}^1(TM,\mathbf{R})$. So (5.32) does not imply $V_\xi^{(t)} \in HV(TM)$ (see(5.22)). Nevertheless, in our situation, we have

Proposition 5.4 *Let $V_\xi^{(t)} \in V(TM)$ as above and $I_\xi^{(t)} \in C^\infty(TM)$ be the function given by (2.18). Then we have*

$$-dI_\xi^{(t)} = \omega(V_\xi^{(t)}, \cdot)\ . \tag{5.34}$$

(*proof*) Putting $V = V_\xi{}^{(t)}$, we have

$$\begin{aligned}
\omega(V,\cdot) &= i(V)\cdot\omega = i(V)\cdot d\theta = \{\mathcal{L}_V - d\cdot i(V)\}\,\theta \\
&\overset{(5.31)}{=} d\Lambda_\xi{}^{(t)} - d\langle\theta, V\rangle \\
&\overset{(5.33)}{=} -dI_\xi{}^{(t)}\,. \qquad\qquad \text{Q.E.D.}
\end{aligned}$$

This proposition means

(5.35) $$V_\xi{}^{(t)} \text{ given by (5.29) } = V_{I_\xi{}^{(t)}} \text{ in the sense of (5.12)}\,,$$

hence $V_\xi{}^{(t)} \in HV(TM)$ and, by (5.30), we have a Lie algebra homomorphism

(5.36) $$\xi \in \mathrm{g} \quad \mapsto \quad V_\xi{}^{(t)} \in HV(TM)\,.$$

We can now prove (ii) in the theorem. In fact we have

$$\left\{I_\xi{}^{(t)}, q^i\right\} = V_{I_\xi{}^{(t)}} \cdot q^i \overset{(5.35)}{=} V_\xi{}^{(t)} \cdot q^i = \left(A_\xi{}^{(t)j}\frac{\partial}{\partial q^j} + B_\xi{}^{(t)j}\frac{\partial}{\partial v^j}\right)\cdot q^i = A_\xi{}^{(t)i}$$

and (2.20b) similarly, proving (ii).

To prove (iii), we choose a basis of g (as a vector space)

(5.37) $$e_1, e_2, \cdots \in \mathrm{g}$$

and, for each e_k, we fix $\Lambda_{e_k}^{(t)} \in C^\infty(TM)$. Then for genaral

(5.38) $$\xi = \sum_k c_k e_k \in \mathrm{g}\,, \qquad c_k \in \mathbf{R}\,,$$

we put

(5.39) $$\tilde{\Lambda}_\xi{}^{(t)} = \sum_k c_k \Lambda_{e_k}^{(t)}\,.$$

It can be seen that (assuming M is connected) the difference between $\tilde{\Lambda}_\xi{}^{(t)}$ and the original $\Lambda_\xi{}^{(t)}$ is constant. We remark that the addition of a constant to $\Lambda_\xi{}^{(t)}$ is admissible, because, in the condition (2.12), $\Lambda_\xi{}^{(t)}$ is concerned only through its derivative. Using this new $\tilde{\Lambda}_\xi{}^{(t)}$ as $\Lambda_\xi{}^{(t)}$ from the first, we have the linear mapping

(5.40) $$\xi \in \mathrm{g} \quad \mapsto \quad I_\xi{}^{(t)} \in C^\infty(TM)\,.$$

Thus, for fixed $t \in \mathbf{R}$, we have the commutative diagram:

$$
\begin{tikzcd}
\xi \in \mathrm{g} \arrow[r, "(5.40)"] \arrow[dr, "(5.36)"'] & C^\infty(TM) \ni I_\xi^{(t)} \arrow[d, "(5.13)"] \\
 & HV(TM) \ni V_\xi^{(t)}
\end{tikzcd}
\tag{5.41}
$$

In the diagram (5.41), the mappings (5.13) and (5.36) are Lie algebra homomorphism but (5.40) is only a linear mapping in general. This is the origin of the appearance of the cocycle term $\alpha^{(t)}(\xi, \eta)$ in (2.27) as shown below.

In this situation, we have

Proposition 5.5 *We fix $t \in \mathbf{R}$. For $\xi, \eta \in \mathrm{g}$, the function on TM*

$$\left\{I_\xi^{(t)}, I_\eta^{(t)}\right\} - I_{[\xi,\eta]}^{(t)} \tag{5.42}$$

is constant. Furthermore the bilinear form $\alpha^{(t)} : \mathrm{g} \times \mathrm{g} \to \mathbf{R}$ defined by

$$\alpha^{(t)}(\xi, \eta) = \left\{I_\xi^{(t)}, I_\eta^{(t)}\right\} - I_{[\xi,\eta]}^{(t)} \tag{5.43}$$

is a cocycle in the sense of (2.25).

(*proof*) Through the homomorphism (5.13), we have, omitting ${}^{(t)}$,

$$\{I_\xi, I_\eta\} \overset{(5.13)}{\longmapsto} V_{\{I_\xi, I_\eta\}} \overset{(5.14)}{=} \left[V_{I_\xi}, V_{I_\eta}\right] \overset{(5.35)}{=} [V_\xi, V_\eta] \overset{(5.30)}{=} V_{[\xi,\eta]},$$

$$I_{[\xi,\eta]} \overset{(5.13)}{\longmapsto} V_{I_{[\xi,\eta]}} \overset{(5.35)}{=} V_{[\xi,\eta]},$$

hence $V_{(5.42)} = 0$, giving $d(5.42) \overset{(5.12)}{=} -\omega(V_{(5.42)}, \cdot) = 0$. Since we have assumed M (hence TM) is connected, we see that (5.42) is a constant.

The argument in p. 44 of [7] gives that (5.43) is a cocycle. Q.E.D.

This proposition yields (iii), proving the whole theorem. Q.E.D.

In the point of view of Theorem 4.2.8 of [1], the relation (5.34) and the linearity of (5.40) mean that the mapping (1.28) is a momentum mapping. A more explicit formula for $\alpha^{(t)}$ of (5.43) is given in Theorem 4.2.8 of [1].

Appendix : Proof of Proposition 5.3

In this appendix, we fix $\xi \in$ g and omit it in notations for simplicity, so $A^{(t)}, \Lambda^{(t)}$ means $A_\xi^{(t)}, \Lambda_\xi^{(t)}$ etc. A similar calculation is done in Appendix of [5], where the proof of the fact corresponding to (ii) of our theorem is given.

The condition (2.19) means

$$A^{(t)i}(q,v) = D^{(t)i}(q) + E^{(t)i}{}_j\, v^j \ , \tag{A.1}$$

which corresponds to (A.2) in [5]. For $\forall q(\cdot) \in \mathrm{Path}\, M$, putting $q^\epsilon(\cdot) = (\exp - \epsilon\xi) \cdot (q(\cdot))$, we have from (2.17)

$$\frac{d}{d\epsilon} q^{\epsilon,i}(t)\bigg|_{\epsilon=0} = A^{(t)i}(q(t), \dot q(t)) \ . \tag{A.2}$$

Thus, for the Lagrangian $L = L(q,v)$ of the form (2.5), we obtain

$$\begin{aligned}
&\frac{d}{d\epsilon} L\left(q^\epsilon(t), \dot q^\epsilon(t)\right)\bigg|_{\epsilon=0} \\
&= \frac{\partial L}{\partial q^i}(q(t),\dot q(t)) \left\{ \frac{d}{d\epsilon} q^{\epsilon,i}(t)\bigg|_{\epsilon=0} \right\} + \frac{\partial L}{\partial v^i}(q(t),\dot q(t)) \left\{ \frac{d}{d\epsilon}\frac{d}{dt} q^{\epsilon,i}(t)\bigg|_{\epsilon=0} \right\} \\
&= -U_i(q(t))\, A^{(t)i}(q(t),\dot q(t)) + \dot q_i(t)\, \frac{d}{dt} A^{(t)i}(q(t),\dot q(t)) \\
&= -U_i(q)\, D^{(t)i}(q) + \left\{ \dot D^{(t)i}(q) - U_j(q)\, E^{(t)ji} \right\} \dot q_i \\
&\quad + \left\{ \dot E^{(t)}{}_{ij} + D^{(t)}{}_{i;j}(q) \right\} \dot q^i \dot q^j + E^{(t)}{}_{ij}\, \dot q^i \ddot q^j \ ,
\end{aligned} \tag{A.3}$$

where $\dot D^{(t)}(q) = \frac{\partial}{\partial t} D(t,q)$, $D^{(t)}{}_{i;j} = \frac{\partial}{\partial q^j} D^{(t)}{}_i$, and we let $i, j, \cdots$ up and down using (a_{ij}) or (a^{ij}) :

$$E_{ij} \equiv a_{ik} E^k{}_j \ , \ E^{ij} \equiv E^i{}_k\, a^{kj} \ , \ \cdots, \ \text{etc.} \tag{A.4}$$

Now we put

$$\Lambda^{(t)}(q,v) = \alpha^{(t)}(q) + \beta^{(t)}{}_i(q)\, v^i + \frac{1}{2}\gamma^{(t)}{}_{ij}(q)\, v^i v^j + [\text{higher terms in } v] \ . \tag{A.5a}$$

We can assume $\left(\gamma^{(t)}_{ij}(q)\right)$ is symmetric :

(A.5b) $$\gamma^{(t)}_{ji}(q) = \gamma^{(t)}_{ij}(q) \ .$$

Then we have

(A.6) $$\frac{d}{dt}\Lambda^{(t)}(q(t),\dot q(t))$$
$$= \dot\alpha^{(t)}(q) + \left\{\alpha^{(t)}_{;i}(q) + \dot\beta^{(t)}_{i}(q)\right\} \dot q^i + \beta^{(t)}_{i}(q)\,\ddot q^i$$
$$+ \left\{\beta^{(t)}_{i;j}(q) + \frac{1}{2}\dot\gamma^{(t)}_{ij}(q)\right\} \dot q^i \dot q^j + \gamma^{(t)}_{ij}(q)\,\dot q^i \ddot q^j$$
$$+ \frac{1}{2}\gamma^{(t)}_{ij;k}(q)\,\dot q^i \dot q^j \dot q^k + [\,\text{terms of } \dot q,\ \ddot q \text{ of order} \geq 3\,] \ .$$

We remark that the condition (2.12) must be satisfied for any $q(\cdot) \in$ Path M, hence the coefficients for $\dot q$, $\ddot q$ in (A.3) and (A.6) must be equal independently. So [higher terms in v] of (A.5a) is absent and we have $\gamma^{(t)}_{ij;k}(q) = 0$ and $\beta^{(t)}_{i}(q) = 0$. Thus we have

(A.7) $$\Lambda^{(t)}(q,v) = \alpha^{(t)}(q) + \frac{1}{2}\gamma^{(t)}_{ij}\,v^i v^j \ , \qquad \gamma^{(t)}_{ji} = \gamma^{(t)}_{ij} \ .$$

We also have

(A.8a) $$-U_i(q)D^{(t)i}(q) = \dot\alpha^{(t)}(q) \ ,$$

(A.8b) $$\dot D^{(t)}_{i}(q) - U_j(q)\,E^{(t)j}_{\ \ i} = \alpha^{(t)}_{;i}(q) \ ,$$

(A.8c) $$\left\{\dot E^{(t)}_{ij} + D^{(t)}_{i;j}(q) - \frac{1}{2}\dot\gamma^{(t)}_{ij}\right\} \dot q^i \dot q^j = 0 \ ,$$

(A.8d) $$E^{(t)}_{ij} = \gamma^{(t)}_{ij} \ .$$

We see, from (A.8d) and (A.5b), that $E^{(t)}_{ij}$ is symmetric:

(A.9) $$E^{(t)}_{ji} = E^{(t)}_{ij} \ .$$

Since (A.8c) is satisfied for all $\dot q$, the symmetrization of i, j in $\{\qquad\}$ must vanish, thus

(A.10) $$D^{(t)}_{i;j}(q) + D^{(t)}_{j;i}(q) + \dot E^{(t)}_{ij} = 0 \ ,$$

here we used (A.8d). Using these relations we have

$$
\begin{aligned}
\text{(A.11)}\quad B^{(t)}{}_i(q,v) &= \frac{d}{dt} A^{(t)}{}_i(q,v) \\
&= \frac{d}{dt}\left[D^{(t)}{}_i(q) + E^{(t)}{}_{ij}\, v^j\right] \\
&= \dot{D}^{(t)}{}_i(q) + D^{(t)}{}_{i;j}(q)\, v^j + \dot{E}^{(t)}{}_{ij}\, v^j + E^{(t)}{}_{ij}\,(-U^j(q)) \\
&= \alpha^{(t)}{}_{;i}(q) - D^{(t)}{}_{j;i}(q)\, v^j \qquad \text{(by (A.8b) and (A.10))}
\end{aligned}
$$

Now we can prove (5.31) as follows. We put

$$
V = V^{(t)} = V_\xi{}^{(t)} = A^{(t)i}\frac{\partial}{\partial q^i} + B^{(t)i}\frac{\partial}{\partial v^i}
$$

and omit $^{(t)}$. Then

$$
\begin{aligned}
\text{(A.12)}\qquad \mathcal{L}_V\,\theta &= \mathcal{L}_V\,(v_i\, dq^i) \\
&= (\mathcal{L}_V\, v_i)\, dq^i + v_i\,\mathcal{L}_V\,(dq^i) \\
&= (V\cdot v_i)\, dq^i + v_i\, d(V\cdot q^i) \\
&= B_i\, dq^i + v_i\, dA^i \\
&= \left[\alpha_{;j}(q) - D_{k;j}\, v^k\right] dq^j \qquad \text{(by (A.11))} \\
&\quad + v_i\left[D^i{}_{;j}(q)\, dq^j + E^i{}_j\, dv^j\right] \\
&= \alpha_{;j}(q)\, dq^j + E_{ij}\, v^i\, dv^j \\
&= d\Lambda\,, \qquad \text{(by (A.7) and (A.8d))}
\end{aligned}
$$

yielding the proposition. Q.E.D.

Acknowledgement. I wish to thank Prof. Y. Yamada for giving us lectures on CFT. This note is an outcome to understand his lectures more deeply. I also wish to thank Profs. T. Shimada, M. Murayama and K. Fujii for fruitful

discussions.

References

1. Abraham, R., Marsden, J.E., *Foundations of mechanics,* Addison–Wesley, Redwood city, 1978.
2. Arnold, V.I., *Mathematical methods of classical mechanics,* Springer, New York, 1989.
3. Green, M.B., Schwartz, J.H., Witten, E., *Superstring theory I,* Cambridge Univ. Press, Cambridge, 1987.
4. Kac, V.G., Raina, A.K., *Highest weight representations of infinite dimentional Lie algebras,* World Scientific, Singapore, 1987.
5. Kugo, T., Ojima, I., *Subsidiary conditions and physical S–matrix unitarity in indefinite–metric quantum gravitation theory,* Nucl. Phys. **B144**, 234 – 252 (1978).
6. Witten, E., *Non–abelian bosonization in two–dimensions,* Commun. Math. Phys. **92**, 455–472 (1984).
7. Woodhouse, N., *Geometric quantization,* Oxford Univ. Press, Oxford, 1980.